Reynolds number dependence of Lyapunov exponents of turbulence and fluid particles

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The Navier-Stokes equations generate an infinite set of generalized Lyapunov exponents defined by different ways of measuring the distance between exponentially diverging perturbed and unperturbed solutions. This set is demonstrated to be similar, yet different, from the generalized Lyapunov exponent that provides moments of distance between two fluid particles below the Kolmogorov scale. We derive rigorous upper bounds on dimensionless Lyapunov exponent of the fluid particles that demonstrate the exponent's decay with Reynolds number Re in accord with previous studies. In contrast, terms of cumulant series for exponents of the moments have power-law growth with Re. We demonstrate as an application that the growth of small fluctuations of magnetic field in ideal conducting turbulence is hyperintermittent, being exponential in both time and Reynolds number. We resolve the existing contradiction between the theory, that predicts slow decrease of dimensionless Lyapunov exponent of turbulence with Re, and observations exhibiting quite fast growth. We demonstrate that it is highly plausible that a pointwise limit for the growth of small perturbations of the Navier-Stokes equations exists.

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I. INTRODUCTION

The Navier-Stokes equations are an infinite-dimensional dynamical system where small perturbations of its solutions grow exponentially. This results in a finite time over which the evolution can be predicted. Thus, for instance, thermal fluctuations change the macroscopic turbulent flow quite quickly [1], the fact that underlies the mechanism via which changes of motion of a single electron can cause global atmospheric changes in a couple of weeks [2].

The growth of small perturbations is traditionally described by a Lyapunov exponent [1,3,4]. The exponent provides the logarithmic growth rate of the distance between the perturbed and unperturbed solutions. The definition of the distance involves introducing a norm in the functional space and it is not obvious which norm must be used. Thus, it is usual to assume that the theorem on the existence and realization-independence of the Lyapunov exponent λ^v, which was proved for finite-dimensional systems [3], generalizes to the Navier-Stokes equations [5–7]. This would tell that $\lim_{t\to\infty} t^{-1}\ln(||\delta\boldsymbol{v}(t)||/||\delta\boldsymbol{v}(0)||)$ is independent of the initial conditions on the perturbation flow $\delta\boldsymbol{v}(t)$ and also of the unperturbed flow. In the case of a finite-dimensional system any definition of the norm $||\delta\boldsymbol{v}(t)||$ would result in the same limit. This however is not necessarily the case for the infinite-dimensional systems where different definitions of the norm can produce different limits. We demonstrate here that for the Navier-Stokes equations $\gamma^v(p) \equiv \lim_{t\to\infty} t^{-1}\ln(||\delta\boldsymbol{v}(t)||_p/||\delta\boldsymbol{v}(0)||_p)$, with the L_p−norms $||\delta\boldsymbol{v}(t)||_p \equiv (\int |\delta v|^p d\boldsymbol{x})^{1/p}$, differ for different p by powers of the Reynolds number Re. Therefore, the usually used definition with the L_2 norm [5–7], giving $\lambda^v = \gamma^v(2)$, leaves outside many essential details of the divergence of the solutions.

There is a controversy in the current knowledge which we propose to be resolved by the different Re dependence of $\gamma^v(p)$ for different p. The theory of Ref. [8] predicts that the dimensionless Lyapunov exponent, obtained by multiplying λ^v with the Kolmogorov timescale [9], decays with Re as a power law with small exponent. However, Ref. [5] observed in direct numerical simulations a power-law growth with an appreciable exponent.

We explain the reason for the discrepancy. The study of λ^v performed in Ref. [8] relies on Ruelle's assumption [1] that the exponent can be estimated as the average of the inverse of the minimal timescale of turbulence. This timescale is given by the local viscous timescale of the flow $t_\nu(\boldsymbol{x})$; see, e.g., Ref. [9]. Roughly the assumption is rooted in the observation that perturbations grow due to local stretching of fluid elements whose rate is determined by the local velocity gradients given by $\sim t_\nu^{-1}(\boldsymbol{x})$. Thus, Ref. [8] assumed $\lambda^v = \langle t_\nu^{-1}\rangle$ where angular brackets stand for averaging. In fact, there is no ground to the last equality because local growth rates fluctuate strongly and it is by no means evident which of them determines the global growth rate of the perturbation λ^v. Indeed, intermittency of turbulence implies that $t_\nu(\boldsymbol{x})$ undergoes strong spatial fluctuations with amplitude proportional to powers of Re, as seen, e.g., from the

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multifractal model [9]. The local growth at the rate given by $t_\nu^{-1}(\boldsymbol{x})$ implies $||\delta\boldsymbol{v}(t)||_p \sim \{\int \exp[pt/t_\nu(\boldsymbol{x})]d\boldsymbol{x}\}^{1/p}$ which reduces to the space average $\langle t_\nu^{-1}\rangle$ only at $p \to 0$ [where $||\delta\boldsymbol{v}(t)||_p \sim (1 + pt\langle t_\nu^{-1}\rangle)^{1/p} \sim \exp(t\langle t_\nu^{-1}\rangle)$] gives $\gamma^v(p=0) = \langle t_\nu^{-1}\rangle$. We remark that technically L_p for $0 < p < 1$ is only a quasinorm however this is irrelevant here and below. It is evident that at $p = 2$, used for calculating λ^v, the averaged quantity is very different from t_ν^{-1}. This results in a power of Reynolds number difference between $p = 2$ and $p \to 0$ cases explaining the observations of Ref. [5]. The above was irrelevant for the Ruelle's work [1] whose purpose was making estimates within the Kolmogorov theory that disregards intermittency.

Dependence of the limit for the Lyapunov exponent on the norm's definition is in sharp contrast with finite-dimensional dynamical systems. There any definition of the distance between the solutions produces the same exponent. Indeed, if an l_p norm of a finite-dimensional vector grows exponentially, then the growth exponent equals the maximal growth exponent of the components. This implies that the growth exponents of all l_p norms are the same. This is not so when the number of degrees of freedom is infinite because the L_p norm depends not only on the local value of the field but also on the space fraction where the value holds, that also depends on time exponentially. As a result, infinite dimensionality of the Navier-Stokes equations demands reconsideration of the facts established for Lyapunov exponents of finite-dimensional systems.

The above considerations raise the issue of whether the limits for the Lyapunov exponent are realization-independent irrespective of which norm is used. If they are, which seems to be the case, then which of them and when must be used. The usage of different p norms would lead to a power of Re difference of the corresponding Lyapunov exponents $\gamma^v(p)$, which can be crucial in astrophysical or other applications with very high Re.

The rationale for using λ^v defined by the L_2 norm is that it describes the difference of energies of perturbed and unperturbed solutions (caution must be exerted though). The energy difference is the sum of $||\delta\boldsymbol{v}||_2^2$ which is quadratic in the perturbation and a nontrivial term which is linear in the perturbation and could dominate the difference. Thus, $||\delta\boldsymbol{v}||_2^2$ could be considered as the lower bound for the energy difference). However, the growth is intermittent and it is possible to have a large energy difference when the local difference of perturbed and unperturbed flows $\delta\boldsymbol{v}(t,\boldsymbol{x})$ is still small in most of the space. Thus, our work indicates that the major fraction of space is described by an exponent very different from λ^v. We demonstrate that it is plausible that the perturbation grows exponentially at almost every spatial point with asymptotically the same growth exponent. This exponent is readily seen to be given by the derivative of $\gamma^v(p)$ at $p = 0$ and differs from λ^v by a power of Re. The difference holds because the asymptotic pointwise growth rate is attained very nonuniformly in space.

Another norm of interest is L_∞ with $\gamma^v(p=\infty)$ providing the growth exponent of the maximal value of the perturbation. In fact, the intermittency of turbulence causes strong inhomogeneity of the perturbation growth. It is characterized by bursts and can only be described by using the infinite set of Lyapunov exponents as given by the full function $\gamma^v(p)$.

The infinite set of Lyapunov exponents $\gamma^v(p)$ is qualitatively similar to another set of Lyapunov exponents associated with the turbulent flow. These exponents also describe the exponential growth of the distance between two infinitesimally close solutions. This time these are solutions of the equation of Lagrangian trajectories. The solutions provide trajectories of fluid particles and form a three-dimensional dynamical system. This system is characterized by a positive Lyapunov exponent λ_1 that describes exponential growth of distance r between two trajectories below the viscous scale [9], the phenomenon often referred to as the Lagrangian chaos; see, e.g., Refs. [10–14].

It was found in Ref. [15] that Re dependence of λ_1, and the dispersion of the finite-time Lyapunov exponent, are described well by the laws predicted in Ref. [8] for λ^v and the dispersion of the finite time Lyapunov exponent of turbulence. The authors have not commented on the reasons of successfully applying the theory for Lyapunov exponents of turbulence to those of fluid particles. This work implies that $\lambda_1 \sim \langle t_\nu^{-1}\rangle$ provides a qualitatively valid description of the Re dependence of the Lyapunov exponent. Indeed, we demonstrate that the calculation of Ref. [8], originally intended for turbulence, in fact, applies to fluid particles, cf. Ref. [16]. This is the reason why using the prediction of Ref. [8] for another quantity, Ref. [15] could explain their observations.

The growth of the distance between two close trajectories of fluid particles is intermittent. There are two different sources of intermittency involved.

The first type of intermittency is well-known, see, e.g., the detailed discussion in Ref. [14]. It does not have anything to do with the intermittency of turbulence. This intermittency originates in randomness of the velocity field and exists even for separation in Gaussian flows that are completely uncorrelated in time, the so-called Kraichnan model [13]. One can understand it in the following way: At large times, much larger than the correlation time of the flow, the most probable value of the finite-time average stretching rate on the trajectory is λ_1 [see Eqs. (5) and (6)]. Thus, $r \sim \exp(\lambda_1 t)$ holds with probability close to one. However, there are also rare trajectories for which the finite-time average stretching rate λ on the trajectory is larger than λ_1. For these trajectories the local velocity gradient is systematically larger than λ_1. This persistence of large gradients occurs not because the separating pair of particles entered a region of long living gradients of the flow. Rather, the gradients in question, that are qualitatively randomly renewed each correlation time, attain atypical value $\simeq \lambda$ after each renewal. The probability of these randomly persistent configurations of the flow decays exponentially in time. Despite the small probability of these rare events, they dominate moments of the inter-particle distance, because they are associated with separation $r \sim \exp(\lambda t)$, which is exponentially larger than $\exp(\lambda_1 t)$.

The other contribution to the intermittency of separation of particle trajectories, which is in fact the dominant contribution at large Re, is due to intermittency of turbulence. The probability of having atypically large persistent velocity gradients, where the finite-time average stretching rate exceeds λ_1 by a power of Re, is significant in turbulence, as opposed to Gaussian randomness. This results in magnification of growth rates by powers of Re, a phenomenon we call "hyperintermittency."

The intermittent separation of the trajectories cannot be described by λ_1 only and demands the introduction of the generalized Lyapunov exponent $\gamma(k)$. This provides the growth exponent of the kth moment of the distance between the trajectories; see, e.g., Ref. [12], for numerical studies and [14] for the theory. We demonstrate that $\gamma(k)/k$ is qualitatively similar to $\gamma^v(k)$ where factor of k is due to insignificant difference in the definitions. This similarity holds since both sets of the exponents are determined by similar processes of local stretching of fluid elements.

The bridge between the two sets of the exponents is provided by the growth rate of small fluctuations of magnetic field in the turbulent flow of a conducting fluid. The field reacts on the transporting turbulent flow via the Lorentz force. We consider only early stages of the magnetic field amplification by the turbulent flow from its infinitesimal seed values where the Lorentz force is negligible. The flow is then prescribed and unaffected by the dynamics of the magnetic field. It obeys the same Navier-Stokes equations as without the magnetic field which is the so-called kinematic dynamo regime [17,18].

The field's growth in ideal conducting flow, defined by setting resistivity to zero, is fully described by $\gamma(k)$ since the magnetic field lines are "frozen" in the fluid [17]. Thus, spatial moment of magnetic field of order k grows in time exponentially with exponent $\gamma(k)$. The long-time asymptotic growth rate at almost every fixed point in space is uniform and given by the Lyapunov exponent λ_1 that equals $\gamma'(0)$. The dimensionless growth exponent $\lambda_1\tau_\nu$, where τ_ν is the Kolmogorov time [9], decays with Re; see above. However, this does not mean that similar dependence holds also for the energy whose dimensionless growth exponent is given by $\gamma(2)\tau_\nu$. We demonstrate that due to intermittency $\gamma(2)\tau_\nu$ grows with Re with appreciable scaling exponent, quite similarly to the growth of $\lambda^v\tau_\nu = \gamma^v(2)\tau_\nu$ observed in Ref. [5]. The difference holds because the energy integral at time t is determined by rare spatial regions where the growth exponent of the energy is larger than λ_1. The volume of these regions shrinks exponentially fast and disappears completely at $t \to \infty$ in accord with the uniform asymptotic growth at exponent λ_1. Still at any finite t this volume is finite and it determines the energy integral, cf. Ref. [14].

The hyperintermittency of separation of trajectories has direct implications for the intermittent growth of the magnetic field. The energy grows due to regions whose volume fraction is exponentially small in time. This is true already in model Gaussian velocity fields; see, e.g., Refs. [18,19]. However, for those fields the rate of growth in these regions is not larger than λ_1 by a power of a large parameter as Re above. It is merely larger than λ_1 resulting in the inequality $\gamma(2) > 2\lambda_1$ where $\gamma(2)/(2\lambda_1)$ is of order one (strictly speaking convexity allows for $\gamma(2) = 2\lambda_1$ however this degenerate case $\gamma(k) = k\lambda_1$ does not seem relevant for random flows). In contrast the growth exponent in the intermittent turbulent flow $\gamma(2)/(2\lambda_1)$ is given by a power of Re and is a larger parameter in the high-Re flows. Thus, due to hyperintermittency the growth is exponentially sensitive both to time and Re. The observation of hyperintermittent growth at large Re poses a formidable challenge due to extreme rarity of the statistically relevant regions.

Finally, before turning to quantitative studies, we describe how the passage from $\gamma(k)$ to $\gamma^v(k)$ can be realized. The data which is available presently and the theory below indicate that $\gamma(k) = c_k \mathrm{Re}^{\beta_k}$, where c_k are either independent or weakly dependent on Re, and β_k is a nontrivial function of k. We consider growth exponents of the magnetic field's moments $\gamma^B(k)$ as functions of magnetic diffusivity η where $\gamma^B(k, \eta = 0) = \gamma(k) = c_k \mathrm{Re}^{\beta_k}$. The Re dependence of $\gamma^B(k, \eta)$ at small but finite η is similar to that of $\gamma(k)$ as can be seen from the solution of Ref. [19] at η much smaller than the kinematic viscosity ν. Thus, though, $\gamma^B(k, \eta)$ has a jump at $\eta = 0$ this jump does not change the Re dependence of the asymptotic growth rate $d\gamma^B/dk(k = 0)$ and does not seem to change qualitatively the Re dependence of other $\gamma(k)$. Furthermore, it does not seem possible that the continuation from small finite η/ν to $\eta/\nu = 1$ will change the Re dependence of $\gamma^B(k)$ qualitatively. Hence, $\gamma^B(k, \eta = \nu)$ depends on Re as a power law with k-dependent exponent. However, the only difference of the equations of $\boldsymbol{B}$ at $\eta = \nu$ and $\delta\boldsymbol{v}$ is a sign of the stretching term and the presence of pressure. Both would not change the qualitative Re dependence proposing that $\gamma^v(k) = b_k \mathrm{Re}^{\delta(k)}$ where Re dependence of b_k is slow.

II. Re DEPENDENCE OF GENERALIZED LYAPUNOV EXPONENT OF FLUID PARTICLES

We assume everywhere in this work that the flow $\boldsymbol{v}(t, \boldsymbol{x})$ evolves according to the Navier-Stokes (NS) equations

$$\partial_t \boldsymbol{v} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\nabla p + \nu\nabla^2\boldsymbol{v} + \boldsymbol{f}, \quad \nabla\cdot\boldsymbol{v} = 0, \tag{1}$$

where p is the pressure divided by the density, ν is the kinematic viscosity and the forcing $\boldsymbol{f}$ ensures stationarity. The flow is characterized by the characteristic value V of velocity at the integral scale L and the Reynolds number $\mathrm{Re} \equiv VL/\nu$. The flow, which is assumed to be turbulent, $\mathrm{Re} \gg 1$, generates a three-dimensional dynamical system whose trajectories $\boldsymbol{q}(t, \boldsymbol{x})$ obey

$$\partial_t \boldsymbol{q}(t, \boldsymbol{x}) = \boldsymbol{v}[\boldsymbol{q}(t, \boldsymbol{x}), t], \quad \boldsymbol{q}(t = 0, \boldsymbol{x}) = \boldsymbol{x}. \tag{2}$$

Here $\boldsymbol{q}(t, \boldsymbol{x})$ are Lagrangian trajectories of the fluid that are labeled by their position at $t = 0$. We consider homogeneous turbulence though many considerations below hold for inhomogeneous turbulence as well. We assume that the flow $\boldsymbol{v}$ is smooth below the viscous scale l_ν. Then the above system is a smooth dynamical system which can be characterized by the Lyapunov exponent. The distance $\boldsymbol{r}(t, \boldsymbol{x}) \equiv \boldsymbol{q}(t, \boldsymbol{x} + \boldsymbol{r}_0) - \boldsymbol{q}(t, \boldsymbol{x})$ between two Lagragian trajectories initially separated by $r_0 \ll l_\nu$ obeys

$$\partial_t \boldsymbol{r}(t, \boldsymbol{x}) = \boldsymbol{v}[\boldsymbol{q}(t, \boldsymbol{x}) + \boldsymbol{r}, t] - \boldsymbol{v}[\boldsymbol{q}(t, \boldsymbol{x}), t] \approx (\boldsymbol{r}\cdot\nabla)\boldsymbol{v}, \tag{3}$$

where $\nabla\boldsymbol{v}$ is evaluated at $\boldsymbol{q}(t, \boldsymbol{x})$ and we consider not too large times such that $r(t) \ll l_\nu$. We concentrate on the distance $r(t)$ by introducing $\boldsymbol{r} = r\hat{n}$ where $|\hat{n}| = 1$. We find from Eq. (3),

$$\frac{d\ln r}{dt} = \hat{n}\nabla\boldsymbol{v}\hat{n}, \quad \frac{d\hat{n}}{dt} = (\hat{n}\cdot\nabla)\boldsymbol{v} - \hat{n}(\hat{n}\nabla\boldsymbol{v}\hat{n}). \tag{4}$$

This gives

$$\frac{1}{t}\ln\left[\frac{r(t,\boldsymbol{x})}{r_0}\right]=\int_0^t \hat{n}(t')\nabla \boldsymbol{v}(t')\hat{n}(t')\frac{dt'}{t}, \tag{5}$$

where we omitted the $\boldsymbol{x}$ dependence in the right-hand side (RHS). The Lyapunov exponent then describes the limit

$$\lim_{t\to\infty}\frac{\ln(r(t,\boldsymbol{x})/r_0)}{t}=\lim_{t\to\infty}\int_0^t \hat{n}\nabla\boldsymbol{v}\hat{n}\frac{dt'}{t}\equiv\lambda_1(\boldsymbol{x}). \tag{6}$$

In the case of time-independent flows the time-average above is described by the Oseledets theorem [3], sometimes called the multiplicative ergodic theorem. Its generalization to random flows, considered here, states that the limit exists and is independent of $\boldsymbol{x}$ and the realization of the velocity [11]. We designate the corresponding constant by λ_1. The independence holds with probability one that is except for set of $\boldsymbol{x}$ with zero total volume (below "for almost all $\boldsymbol{x}$" or "almost everywhere," abbreviated as a.e.) and for velocity fields with zero total probability measure. Here the probability is defined by the so-called natural measure which corresponds to averaging over both $\boldsymbol{x}$ and the realization of the flow [11]. Qualitatively, similarly to the usual ergodic theorem, the multiplicative ergodic theorem means that $\hat{n}\nabla\boldsymbol{v}\hat{n}$ can be considered as a stationary random process with a finite correlation time so that the law of large numbers holds.

We observe that since $\lambda_1(\boldsymbol{x})=\lambda_1$ with probability one then λ_1 is a self-averaging quantity. It can be obtained by averaging Eq. (6) over $\boldsymbol{x}$ and velocity statistics. We start from space averaging

$$\lambda_1=\lim_{t\to\infty}\frac{1}{t}\int d\boldsymbol{x}\int_0^t dt'\hat{n}\nabla\boldsymbol{v}\hat{n}, \tag{7}$$

where the volume was set to one by choice of units of length. We observe that $\hat{n}(t)$ relaxes exponentially for almost all $\boldsymbol{x}$ to a unique direction that is independent of the initial condition $\hat{n}(0)$. The characteristic relaxation time is few τ_ν; see, e.g., Ref. [20] for detailed consideration. This unique direction, called below the major stretching direction, defines a field $\hat{n}(t,\boldsymbol{x})$. This field can be defined at any t by considering long evolution of $\hat{n}$ on Lagrangian trajectories that arrive at $\boldsymbol{x}$ at time t, or, more directly, by studying backward in time evolution. Since we consider infinite-time limit while $\hat{n}$ settles on the major stretching field $\hat{n}(t,\boldsymbol{x})$ after few Kolmogorov times τ_ν then

$$\lambda_1=\lim_{t\to\infty}\frac{1}{t}\int d\boldsymbol{x}\int_0^t dt'[\hat{n}\nabla\boldsymbol{v}\hat{n}][t',\boldsymbol{q}(t',\boldsymbol{x})]$$
$$=\lim_{t\to\infty}\frac{1}{t}\int d\boldsymbol{x}\int_0^t dt'\hat{n}(t',\boldsymbol{x})\nabla\boldsymbol{v}(t',\boldsymbol{x})\hat{n}(t',\boldsymbol{x}), \tag{8}$$

where we changed integration variable from $\boldsymbol{x}$ to $\boldsymbol{q}(t',\boldsymbol{x})$ keeping the notation for the integration variable with no ambiguity. This quantity by the property of $\lambda_1(\boldsymbol{x})$ described above must be independent of the realization of the flow if the realization is typical. Indeed, the space-time average in Eq. (8) is of the type which is known to be independent of the flow. This is often used in turbulence studies for numerical calculations of averages over ensemble of velocities by using space-time averaging instead; see Ref. [21] and references therein. Thus, the above representation allows to obtain λ_1 numerically by using only one realization of the flow. One needs to average $\hat{n}\nabla\boldsymbol{v}\hat{n}$ at random points in space and time where here and below $\hat{n}$ is understood as the local major stretching direction.

Since the last term in Eq. (8) is independent of the velocity realization, averaging this equation over the velocity ensemble we find the representation of λ_1 as velocity average, $\lambda_1=\langle\hat{n}\nabla\boldsymbol{v}\hat{n}\rangle$. Here the remaining space-time averaging can be omitted since the velocity average is a constant in space and time stationarity and spatial homogeneity of turbulence. Here and below the angular brackets without subscript stand for the velocity ensemble average.

We demonstrate how the representation of λ_1 as velocity average of $\hat{n}\nabla\boldsymbol{v}\hat{n}$, with $\hat{n}$ the major stretching direction, could be also obtained by first averaging over the velocity and then over space. We have after averaging Eq. (6) over the velocity

$$\lambda_1=\lim_{t\to\infty}\int_0^t\langle\hat{n}\nabla\boldsymbol{v}\hat{n}\rangle\frac{dt'}{t}=\lim_{t\to\infty}\langle\hat{n}(t)\nabla\boldsymbol{v}(t)\hat{n}(t)\rangle, \tag{9}$$

where we use that at large times the process $\hat{n}\nabla\boldsymbol{v}\hat{n}$ is stationary due to relaxation of $\hat{n}$ to the major stretching direction and the average is constant. Since the ensemble average above is independent of $\boldsymbol{x}$ for homogeneous turbulence then further averaging over $\boldsymbol{x}$, that must be done in principle [11], is redundant. Thus, we recover that spatio-temporal average in Eq. (8) must be equal to velocity ensemble average $\langle\hat{n}\nabla\boldsymbol{v}\hat{n}\rangle$ as necessary for the self-consistency of the assumption made after Eq. (8).

The main conclusion from the above that is needed below is that λ_1 equals the average over the velocity ensemble $\langle\hat{n}\nabla\boldsymbol{v}\hat{n}\rangle$ where $\hat{n}$ is the major stretching direction. For turbulence λ_1 is positive, the fact that defines Lagrangian chaos of motion of fluid particles.

A. Dependence of the Lyapunov exponent on Re

We consider the Reynolds number dependence of the dimensionless Lyapunov exponent $\lambda_1\tau_\nu$ where $\tau_\nu\equiv\sqrt{\nu/\epsilon}$, defined by the energy dissipation rate per unit volume ϵ, is the Kolmogorov time [9]. Using incompressibility and spatial homogeneity we write

$$\frac{\epsilon}{\nu}=\langle(\nabla_i u_k)(\nabla_i u_k)\rangle=\langle(\nabla_i u_k+\nabla_k u_i)\nabla_i u_k\rangle=2\langle\mathrm{tr}s^2\rangle,$$

where $s_{ik}\equiv(\nabla_i v_k+\nabla_k v_i)/2$ is the rate-of-strain matrix. It should be emphasized that τ_ν introduced here is a parameter that characterizes the entire flow and known as the Kolmogorov time, and not the fluctuating local viscous time t_ν considered in the Introduction.

1. Inequalities on λ_1

We introduce the ordered eigenvalues s_i of the symmetric rate-of-strain matrix, $s_1\geqslant s_2\geqslant s_3$. The incompressibility condition implies that $s_1+s_2+s_3=0$ and therefore that $s_1\geqslant 0$ and $s_3\leqslant 0$. The first inequality may be derived from the relationship $\lambda_1=\langle\hat{n}s\hat{n}\rangle$, which yields

$$\lambda_1^2\leqslant\left\langle(\hat{n}s\hat{n})^2\right\rangle\leqslant\left\langle\max_{i=1}^3 s_i^2\right\rangle. \tag{10}$$

We observe that $|s_2|\leqslant s_1$ and $|s_2|\leqslant|s_3|$. Thus, $\max_i s_i^2$ is either s_1^2 or s_3^2. If $\max_i s_i^2=s_1^2$, then $s_2<0$ and $s_2^2+s_3^2\geqslant s_1^2/2$. Here we rely on the result that the minimum of $s_2^2+s_3^2$, subject to the incompressibility condition $|s_2|+|s_3|=s_1$ is attained at $|s_2|=|s_3|=s_1/2$. This results in $\mathrm{tr}s^2=\sum_{i=1}^3 s_i^2 \geqslant 3s_1^2/2=3\max_i s_i^2/2$. Performing similar consideration in the case where $\max_i s_i^2=s_3^2$ by sign reversal of s_i we conclude that

$$\frac{3\max_{i=1}^3 s_i^2}{2}\leqslant \mathrm{tr}s^2. \tag{11}$$

We find by comparing with ϵ/ν the inequality

$$\lambda_1^2\leqslant \langle \max_i s_i^2\rangle \leqslant \frac{2\langle \mathrm{tr}s^2\rangle}{3}=\frac{\epsilon}{3\nu}, \tag{12}$$

or

$$\lambda_1\tau_\nu\leqslant \frac{1}{\sqrt{3}}=0.577, \tag{13}$$

cf. Ref. [22]. For Re that are accessible by today's simulations it is found that $\lambda_1\tau_\nu$ is about 0.14; see Ref. [15] and cf. Ref. [12]. The value of the above inequality is that it demonstrates that $\lambda_1\tau_\nu$ is bounded from above and cannot grow with the Reynolds number indefinitely, as by a power law or otherwise; see below.

A second and stronger inequality may be derived by writing $\lambda_1\leqslant\langle|\hat{n}s\hat{n}|\rangle\leqslant\langle\max_{i=1}^3|s_i|\rangle$. Since $\max_{i=1}^3|s_i|=\sqrt{\max_i s_i^2}$, we find from Eq. (11) that

$$\lambda_1\tau_\nu\leqslant\frac{\langle\sqrt{\tilde{\epsilon}}\rangle}{\sqrt{3}},\quad \tilde{\epsilon}\equiv\frac{\mathrm{tr}s^2}{\langle\,\mathrm{tr}s^2\rangle}, \tag{14}$$

where we introduced the normalized dissipation $\tilde{\epsilon}$. This inequality allows to derive a power-law decay of $\lambda_1\tau$ with Re if we can make the usual assumption on the power-law dependence of the moments of dissipation on Re caused by the intermittency,

$$\langle\tilde{\epsilon}^k\rangle\sim \mathrm{Re}^{\sigma(k)}; \tag{15}$$

see, e.g., Refs. [9,21,23]. It is readily seen from Hölder's inequality that $\sigma(k)$ is convex. Then the constraints $\sigma(0)=\sigma(1)=0$, implied by the definition in Eq. (15) mean that $\sigma(k)$ is nonpositive for $0<k<1$ and nonnegative otherwise [21]. Thus, $\sigma(1/2)\leqslant 0$ and, assuming that there is no degeneracy and the inequality is strict, $\sigma(1/2)<0$, we find from Eq. (14) that $\lambda_1\tau_\nu$ is bounded from above by a power-law function of Re with negative exponent.

It is highly plausible that the bound holds as an order of magnitude equality so that

$$\lambda_1=\langle\hat{n}\nabla \boldsymbol{v}\hat{n}\rangle\sim\langle\sqrt{(\nabla \boldsymbol{v})^2}rangle,\quad \lambda_1\tau_\nu=c_0\mathrm{Re}^{\sigma(1/2)}, \tag{16}$$

where c_0 is either a constant or a function that depends on Re slower than a power law. For simple approximate calculation of $\sigma(1/2)$, see Ref. [21].

2. *Prediction of multifractal model*

In the frame of phenomenology of turbulence [9] velocity gradients are estimated as t_ν^{-1} where t_ν is the fluctuating viscous time introduced before. We find then $\lambda_1=\langle\hat{n}\nabla \boldsymbol{v}\hat{n}\rangle\sim\langle t_\nu^{-1}\rangle$. The last average was calculated using the multifractal model [9] in Ref. [8], yielding

$$\lambda_1\tau_\nu\sim\mathrm{Re}^{-\delta}, \tag{17}$$

with $\delta\simeq 0.041$ (below we often refer to the equation above where δ is considered as a phenomenological constant, not necessarily equal to 0.041). This prediction, understood as the prediction for λ_1 and not λ_1^v, was confirmed qualitatively in Ref. [15]. The work observed slow robust decay of $\lambda_1\tau_\nu$ with the Taylor microscale Reynolds number Re_λ taking values of 65, 105, and 185. Therefore, we seem to have all indications that $\lambda_1\tau_\nu$ decays with Re as a power law, albeit with a small exponent. This smallness is significant because Re could get very large in applications. If $\delta=0.041$ of Ref. [8] is accepted in Eq. (17), then $\lambda_1\tau_\nu$ is of order 10^{-1} at all practically relevant Re, including Re as high as 10^{15} appearing in astrophysical applications.

We explained in Sec. I that Ref. [8] assumed $\lambda^v\sim\langle t_\nu^{-1}\rangle$. Correspondingly, they predicted $\lambda^v\tau_\nu\sim\mathrm{Re}^{-\delta}$ which as we discussed contradicts the observations. Our consideration above shows that by itself the calculation of Ref. [8] is useful.

3. *Remarks*

The above derivations assume multifractality, intermittency, and breakdown of the Kolmogorov 1941 theory's posit of self-similarity of turbulence. These assumptions are confirmed by absolute majority of the measurements that exist today; see, e.g., the references in Refs. [9,23] for Eq. (15) with $\sigma(k)\neq 0$. However, there exists an alternative view that Eq. (15) is only a finite Re effect and at larger Re at least some moments of $\tilde{\epsilon}$ become constants, independent of Re, much in agreement with the Kolmogorov theory [24]. This view seemingly would imply that $\lambda_1\tau_\nu\sim\mathrm{Re}^{-\delta}$ is only an intermediate law applying at finite Re and $\lim_{\mathrm{Re}\to\infty}\lambda_1\tau_\nu$ is finite. Further the derivation of $\lambda_1\tau_\nu\sim\mathrm{Re}^{-\delta}$ by Ref. [8] could also be criticized since it uses the phenomenology of turbulence that was criticized recently in Ref. [25] who propose that the phenomenology does not hold at least until the Taylor-microscale Reynolds number of order 10^4. However, Ref. [25] would seemingly confirm Eq. (15) so that our derivation would work. We conclude that it is highly plausible that $\lambda_1\tau_\nu\sim\mathrm{Re}^{-\delta}$ with a small, nonzero δ is valid.

Qualitatively the decay of $\lambda_1\tau_\nu$ with Re holds because quiescent regions of turbulence, where chaos is depleted, become longer in time and larger in space due to intermittency [9]. The increase of quiescent regions is accompanied by increase in the amplitude of the bursts. However, it is the former that determine λ_1.

4. *Other Lyapunov exponents*

The behavior of λ_1 has implications for other Lyapunov exponents λ_i. The Lyapunov exponents of the fluid particles in the dissipation range of turbulence, where the ordering $\lambda_i\geqslant\lambda_{i+1}$ is assumed, are defined so that $\lambda_1+\lambda_2$ is the logarithmic growth rate of infinitesimal area elements and $\lambda_3=-\lambda_1-\lambda_2$ (more generally, $\sum_{i=1}^3\lambda_i$ is the logarithmic rate of growth of infinitesimal volumes which vanishes for incompressible flows; see Refs. [13,26] for definitions). We observe that $\lambda_3=-\lambda_1-\lambda_2$ and $\lambda_i\geqslant\lambda_{i+1}$ give the inequality $-2\lambda_1\leqslant\lambda_3\leqslant-\lambda_1/2$ that implies that λ_3 (and thus also λ_2) must obey similar asymptotic dependence on the Reynolds number (presently available simulations reveal that $\lambda_2\simeq\lambda_1/4$ and $\lambda_3\simeq-5\lambda_1/4$; see, e.g., Ref. [12]). This conclusion is also seen from the observation that dependence of λ_3 on the statistics of velocity gradients is very similar to that of λ_1; see Appendix of Ref. [10].

B. Generalized Lyapunov exponent

We saw above that the dimensionless first Lyapunov exponent, that describes logarithmic growth rate of $r(t,\boldsymbol{x})$, depends on Re in observable yet not that strong way. Here we demonstrate that similar dependence for the growth exponent of other moments of $r(t,\boldsymbol{x})$ can be strong. These moments are physically relevant, e.g., the growth exponent of $\langle r^2(t)\rangle$ describes the growth of magnetic energy that will be described later and is also more similar to λ^v than λ_1. Here we consider growth exponent $\gamma(k)$ of the kth moment and describe its dependence on Re (here k can assume any real value not just integer values).

1. Definition and properties of generalized Lyapunov exponent

We define the so-called generalized Lyapunov exponent $\gamma(k)$. This exponent describes the growth of moments of the distance between two infinitesimally close trajectories and can be introduced via

$$\gamma(k)=\lim_{t\to\infty}\frac{\ln\left(\langle r^k(t)\rangle_s/r_0^k\right)}{t}, \tag{18}$$

where the limit $r_0\to 0$ is assumed to be taken before $t\to\infty$ so that $r(t)\ll l_\nu$ at all relevant t; cf. Ref. [12]. Equivalently, $r(t)=r_0\exp\left(\int_0^t\hat{n}\nabla\boldsymbol{v}\hat{n}dt'\right)$ is used in the above equation; see Eq. (5). The subscript s next to the angular brackets in Eq. (18) denotes spatial averaging over $\boldsymbol{x}$ so that

$$\langle r^k(t)\rangle_s\equiv\int r^k(t,\boldsymbol{x})d\boldsymbol{x}=r_0^k\left\langle\exp\left(k\int_0^t\hat{n}\nabla\boldsymbol{v}\hat{n}dt'\right)\right\rangle_s \sim r_0^k\exp[\gamma(k)t]. \tag{19}$$

The asymptotic equality holds at times much larger than the correlation time of the stationary process $\hat{n}\nabla\boldsymbol{v}\hat{n}$, attained after $\hat{n}(t)$ relaxes to the major stretching direction; see below.

It is by no means obvious that the limit in Eq. (18) gives a quantity that is independent of the realization of the flow. Indeed, the general formalism of random flows [11] instructs us to average both over space and the ensemble of the velocities. For instance, the spatial average $\langle r^2(t)\rangle_s\equiv\int r^2(t,\boldsymbol{x})d\boldsymbol{x}$ at finite t will vary from realization to realization of the flow. This is so because a given realization of the velocity field cannot produce all possible values of the RHS of Eq. (5) in finite time. However, in the infinite time limit a given realization will almost surely scan through all possible values of the RHS of Eq. (5). Thus, we anticipate that $\gamma(k)$ as defined in Eq. (18) should be independent of the realization of the flow. We shall now provide strong empirical arguments in favor of this realization-independence:

2. Cumulant series and realization independence of $\gamma(k)$

A representation of $\gamma(k)$ as series in cumulants was introduced in Ref. [14]. Using the cumulant expansion theorem [27] we obtain

$$\gamma(k)=\lim_{t\to\infty}\frac{1}{t}\ln\left\langle\exp\left(k\int_0^t\hat{n}\nabla\boldsymbol{v}\hat{n}dt'\right)\right\rangle_s=k\lambda_1+\sum_{n=2}^{\infty}\frac{k^n}{n!}\lim_{t\to\infty}\frac{1}{t}\times\int_0^t\langle\hat{n}\nabla\boldsymbol{v}\hat{n}(t_1)\hat{n}\nabla\boldsymbol{v}\hat{n}(t_2)\dots\hat{n}\nabla\boldsymbol{v}\hat{n}(t_n)\rangle_s^c\prod_{i=1}^n dt_i, \tag{20}$$

where $\hat{n}$ is the major stretching direction and the superscript c stands for cumulant. We remind the reader that the cumulant of the product of n random variables, such as $\hat{n}\nabla\boldsymbol{v}\hat{n}(t_i)$ above, is defined by taking out of the average all correlations of lower order. Thus, for three random variables x, y, and z we have $\langle xyz\rangle^c=\langle xyz\rangle-\langle x\rangle\langle yz\rangle-\langle y\rangle\langle xz\rangle-\langle z\rangle\langle xy\rangle+2\langle x\rangle\langle y\rangle\langle z\rangle$. This implies that the cumulant in the integrand of the last line of the above equation is appreciable only when all t_i are close within the correlation time of $\hat{n}\nabla\boldsymbol{v}\hat{n}(t)$, see details in Ref. [27].

We see that $\gamma(k)$, the infinite-time limit of the cumulant generating function, is given by a series whose terms are space-time averages quite similar to the second average in Eq. (8). Following the same logic as we did after Eq. (8), these terms can be taken as realization-independent. These averages can be assumed [14] to be equal to the averages over the ensemble of velocities according to

$$\gamma(k)=k\lambda_1+\frac{k^2\mu}{2}+\sum_{n=3}^{\infty}\frac{k^n\Delta_n}{n!},$$
$$\Delta_n\equiv\int_{-\infty}^{\infty}\langle\hat{n}\nabla\boldsymbol{v}\hat{n}(0)\hat{n}\nabla\boldsymbol{v}\hat{n}(t_1)\dots\hat{n}\nabla\boldsymbol{v}\hat{n}(t_{n-1})\rangle^c\prod_{i=1}^{n-1}dt_i. \tag{21}$$

We introduced in Eq. (21) the standard notation for the second-order cumulant [8,15]

$$\mu=\int_{-\infty}^{\infty}\langle[\hat{n}\nabla\boldsymbol{v}\hat{n}(0)-\lambda_1][\hat{n}\nabla\boldsymbol{v}\hat{n}(t)-\lambda_1]\rangle dt. \tag{22}$$

Thus, we see that indeed $\gamma(k)$ is independent of velocity realization because it provides growth accumulated over infinite time. This is so because space averaging is always accompanied in Eq. (20) by time-average, in contrast to $\langle r^k(t)\rangle_s$.

We shall demonstrate in Sec. III that $\gamma(k)$, for any fixed k, can be considered as a Lyapunov exponent of an infinite-dimensional dynamical system consisting of the magnetic field in ideally conducting fluid. The k dependence of the Lyapunov exponent arises because of different norms in functional space, in much the same way as $\gamma^v(p)$ which we discussed in the Sec. I. We see that also in this interpretation $\gamma(k)$, as a Lyapunov exponent, is a realization-independent quantity.

To summarize, $\gamma(k)$ in Eq. (18) is well-defined. In particular, we mention its salient properties (see Ref. [14] and references therein): This function is convex, has two zeros at $k=-3$ and $k=0$, is negative for $-3<k<0$ and positive elsewhere. Note also $\gamma'(0)=\lambda_1>0$ and $\gamma'(-3)=\lambda_3<0$.

3. Dispersion μ

Multiplication of the series in Eq. (21) by τ_ν renders all terms dimensionless. In addition, they are nontrivial functions of Re. The most well-studied quantity, besides $\lambda_1\tau_\nu$ discussed above, is $\mu\tau_\nu$. In contrast to $\lambda_1\tau_\nu$, it is a growing function of Re. It was found in Ref. [15] that $\mu\tau_\nu$ depends on Re significantly stronger than $\lambda_1\tau_\nu$. These authors observed that this result agrees with the calculation of the RHS of Eq. (22) by Ref. [8] that was done by using the shell model of turbulence (it must be remarked though that Ref. [8] interpreted this quantity not as the dispersion μ but rather as analogous dispersion describing the growth of distance between solutions of the Navier-Stokes equations). The simulations of Ref. [8] gave $\mu\tau_\nu \sim \mathrm{Re}^\kappa$ with $\kappa \sim 0.3$. These authors elucidated the reason for this rather strong growth of μ with Re. They argue that the equal-time correlation function in the integrand of Eq. (22) is not influenced by intermittency. It is proportional to energy dissipation divided by the viscosity and thus scales as Re (up to the weak Re dependence of $\lambda_1\tau_\nu$, that enters the definition of μ, that can be disregarded). Therefore, the dependence of $\mu\tau_\nu$ on Re arises due to long correlation times of moderate gradients. Due to intermittency the periods of calm turbulence become longer as Re grows so that the correlation time behaves as a power of Re leading to $\mu\tau_\nu \sim \mathrm{Re}^\kappa$ law. Thus, $\mu\tau_\nu$ grows as a power of Re not because the local gradients exceed the typical value, but rather because the typical gradients are persistent over very long times.

It must be said that we are in a much worse position for estimating μ theoretically as compared to λ_1. As was discussed above we can write $\lambda_1 \sim \langle t_\nu^{-1}\rangle$ where $\langle t_\nu^{-1}\rangle$ can be estimated using the multifractal model. There is no multifractal or similar modeling that would work for μ. The reason is that the multifractal model assumes validity of the phenomenology of turbulence [9]. Within that phenomenology the lifetime of a flow configuration with a given local stretching rate $|\hat{n}\nabla\boldsymbol{v}\hat{n}|$ is given by the inverse of that rate (as estimated from taking derivative of the Navier-Stokes equations giving $\partial_t\nabla\boldsymbol{v} = -(\nabla\boldsymbol{v})^2 + \ldots$). This would in fact predict that μ in Eq. (22) behaves roughly as $|\hat{n}\nabla\boldsymbol{v}\hat{n}|$ and is similar to λ_1, which it is not. An adequate description requires a refinement of the phenomenology which also incorporates large fluctuations of lifetimes of configurations with a given $|\hat{n}\nabla\boldsymbol{v}\hat{n}|$.

4. *Relevance of our results for numerical computation of cumulants*

For concreteness, let us demonstrate this relevance for the computation of μ. It is preferable to use spatiotemporal averaging, which is more feasible practically than averaging over the velocity ensemble. We see from Eq. (20) that after finding λ_1 we can obtain μ as the $t \to \infty$ limit of

$$\mu = \int d\boldsymbol{x} \int_0^t \{\hat{n}\nabla\boldsymbol{v}\hat{n}[t_1, \boldsymbol{q}(t_1, \boldsymbol{x})] - \lambda_1\} \times \{\hat{n}\nabla\boldsymbol{v}\hat{n}[t_2, \boldsymbol{q}(t_2, \boldsymbol{x})] - \lambda_1\} \frac{dt_1 dt_2}{t}. \tag{23}$$

If we ignore intermittency, then the limit converges at rather small t. We observe that the value of $\hat{n}(0, \boldsymbol{x})$ is determined by velocity gradients on the trajectory $\boldsymbol{q}(t, \boldsymbol{x})$ [with $t < 0$ in Eq. (2)] within the preceding time interval of order τ_ν (that is $-\tau_\nu \lesssim t < 0$). Here τ_ν is the relaxation time of $\hat{n}(t)$ as we explained previously. This implies that the correlation length of $\hat{n}(0, \boldsymbol{x})$ is the viscous scale l_ν, since this is the scale over which $\nabla\boldsymbol{v}[t, \boldsymbol{q}(t, \boldsymbol{x})]$, with $-\tau_\nu < t < 0$, changes as a function of $\boldsymbol{x}$. Since the correlation scale of velocity gradients is also the viscous scale then we conclude that l_ν is the characteristic scale of variations of $\hat{n}\nabla\boldsymbol{v}\hat{n}(t, \boldsymbol{x})$.

Exponential separation implies that trajectories that stay within the correlation length l_ν from each other during time interval t must be initially separated by distance of order $l_\nu \exp(-\lambda_1 t)$ or smaller. In other words for $t \gg \tau_\nu$ the integrand of the spatial integral in Eq. (23) varies over a characteristic scale $l_\nu \exp(-\lambda_1 t)$. Therefore, effectively the space average is carried over roughly $[L/l_\nu \exp(-\lambda_1 t)]^3$ independent random variables. Here L is the system size which is at least the integral scale so that $(L/l_\nu)^3 \gtrsim \mathrm{Re}^{9/4}$. The number of independent random variables grows exponentially and quickly gets so large that the law of large numbers applies. The RHS of Eq. (23) becomes then time-independent and provides μ.

The inclusion of intermittency is necessary because, as we saw previously, μ is determined by quiescent eddies whose correlation time is proportional to a power of Re. As a result the RHS of Eq. (23) will only become time independent at $t \sim \tau_\nu \mathrm{Re}^\rho$ with some ρ numerically close to $\kappa \sim 0.3$ above. It is possible to rewrite Eq. (23) in the form that demonstrates that μ is time average of time-integrated spatial correlation function

$$\mu = 2\int_0^t \frac{dt_1}{t} \int_{t_1}^t dt_2 \int d\boldsymbol{x}'[\hat{n}\nabla\boldsymbol{v}\hat{n}(t_1, \boldsymbol{x}') - \lambda_1] \times \{\hat{n}\nabla\boldsymbol{v}\hat{n}[t_2, \boldsymbol{q}(t_2|t_1, \boldsymbol{x})] - \lambda_1\}, \tag{24}$$

where $\boldsymbol{x}' = \boldsymbol{q}(t_1, \boldsymbol{x})$ in Eq. (23). We introduced Lagrangian trajectories that depend on initial time

$$\partial_t \boldsymbol{q}(t|t', \boldsymbol{x}) = \boldsymbol{v}(\boldsymbol{q}(t|t', \boldsymbol{x}), t), \quad \boldsymbol{q}(t = t'|t', \boldsymbol{x}) = \boldsymbol{x}, \tag{25}$$

cf. Eq. (2). Similar considerations can be made for higher order cumulants.

5. *Exponent as series in powers of* Re

Phenomenology of turbulence [9] implies that the cumulants behave as powers of Re so that Δ_n in Eq. (21) obey

$$\tau_\nu \Delta_n = c_n \mathrm{Re}^{\beta_n}, \tag{26}$$

with a certain function β_n and dimensionless functions c_n that are either constants or depend on Re slower than a power law. The previously considered case of μ with $n = 2$ is special. The equal time correlation function, obtained by setting $t_1 = t_2$ in the integrand of Eq. (24), is given by the square of velocity gradients that is not influenced by intermittency and whose average is of order ϵ/ν. The nontrivial Re dependence of μ comes from the size of the effective integration range in $t_2 - t_1$ variable, the correlation time. In contrast, for $n \geqslant 3$ already the equal-time correlation functions in Δ_n are influenced by intermittency and have nontrivial Re dependence. These functions include moments of velocity gradients of order higher than two, that are due to intermittent bursts and are determined by velocity gradients larger than τ_ν^{-1} by a power of Re, cf. Eq. (15).

In the frame of the traditional phenomenology of turbulence [9], however, the correlation time of gradients s_l is s_l^{-1}, cf. the discussion of lifetime of the stretching rate above.

Thus, large gradients have small correlation times. This implies that, despite that equal time-correlation functions in Δ_n are formed by events with very large gradients, the contribution of these events into the time integral defining Δ_n could be negligible. This is because time integration approximately multiplies all gradients s_l in Δ_n, but one, by small correlation time $1/s_l$.

The above consideration demonstrates that if the lifetime of large gradients in turbulence can be estimated as their inverse, and yet moderate gradients' lifetime can be much larger than their inverse, then $\Delta_{n\geqslant 3}$ can be determined by the moderate gradients. At the same time, it is plausible [28] that long-living vortices with large vorticity are associated in the turbulent flow with large strains (which are more relevant than vorticity regions since the trajectories separate predominantly in strain regions) whose correlation time exceeds the inverse strain by a power of Re, cf. Ref. [25]. Thus, further studies are necessary to determine which events determine Δ_n and ultimately $\gamma(k)$ given by

$$\gamma(k)\tau_\nu = c_1\mathrm{Re}^{-\delta}k + \frac{c_2\mathrm{Re}^{\kappa}k^2}{2} + \sum_{n=3}^{\infty}\frac{c_n\mathrm{Re}^{\beta_n}k^n}{n!}. \tag{27}$$

The lack of knowledge of how c_n and β_n depend on n does not allow to fix the functional form of the dependence of $\gamma(k)$ on Re. It is plausible that in the limit $\mathrm{Re}\to\infty$ (which could require Re higher than those in most applications) one term determines the whole sum and then $\gamma(k)$ also obeys a power-law dependence on Re. (We remark that truncation of the series for $\gamma(k)$ does not contradict Pawula's theorem [14].) This assumption is made stronger by the observations of Ref. [5] for the Lyapunov exponent of turbulence λ_1^v, considered in more detail later. We demonstrate below that λ_1^v is similar to $\gamma(2)$. Since the former is observed to obey power-law dependence in Re, then a similar dependence would hold for $\gamma(2)$ [and then also for other $\gamma(k)$]. Further indication of power-law behavior is obtained by considering high Re where the first term in the series in Eq. (27) is negligible. There we have $\gamma(k)\approx c_2\mathrm{Re}^{\kappa}k^2/2$ for k small (yet not as small as $\lesssim\mathrm{Re}^{-\kappa-\delta}$). Then, if we can assume that, e.g., for $k=0.1$ the last equality holds uniformly in Re, then the monotonic increase of $\gamma(k)$ with k would imply that $\gamma(2)$ is bounded from below by Re^{κ} times a (small) constant. The last very plausible demonstration that $\gamma(k)$ must obey power-law behavior in Re comes from the large deviations theory considered below.

C. Large deviations and hyperintermittency

The dependence of the moments of $r(t)$ on time, described by Eq. (19) implies that the probability density function of $\rho(t,\boldsymbol{x})\equiv t^{-1}\ln(r(t,\boldsymbol{x})/r_0)$ is described by the large deviations theory [9,14]

$$P(\rho,t)\sim e^{-tS(\rho)},\quad \langle r^k(t)\rangle_s\sim\int e^{t(k\rho-S(\rho))}d\rho, \tag{28}$$

where $S(\rho)$ is the so-called large deviations function. At large times the integral in Eq. (28) is determined by the maximum of the exponent demonstrating that $\gamma(k)$ and $S(\rho)$ form a Legendre transform pair, $\gamma(k)=\max_\rho[k\rho-S(\rho)]$. It is readily seen from the properties of the Legendre transformation that $S(\rho)$ is a convex nonnegative function that has a unique minimum of zero at $\rho=\lambda_1$. Thus, $S(\rho)$ describes the volume fraction (recall that the averages are spatial) of regions where the asymptotic equality $\ln r(\boldsymbol{x},t)/t\approx\lambda_1$ is violated significantly at however large t. The volume of these regions decays exponentially thus leading to $\lim_{t\to\infty}P(\rho,t)=\delta(\rho-\lambda_1)$, implied by $\exp(-tS(\rho))$ form. This is equivalent to the statement that $\lambda_1(\boldsymbol{x})$ in Eq. (6) equals λ_1 for almost all $\boldsymbol{x}$.

Intermittency implies that $\gamma(k)$ is nonlinear, and that the maximum of $k\rho-S(\rho)$ is attained at some $\rho=q(k)$ which is different from λ_1. This signifies that extremely rare regions of space, whose volume fraction $\sim\exp\{-tS[q(k)]\}$ is exponentially small at large times, determine $\langle r^k\rangle_s$. This happens because $r(t,\boldsymbol{x})\sim\exp[q(k)t]$ grows there anomalously fast, see above and, e.g., detailed discussion in Ref. [14]. It is seen from the definition $\rho(t,\boldsymbol{x})\equiv t^{-1}\int_0^t\hat{n}\nabla\boldsymbol{v}\hat{n}dt'$ that qualitatively $q(k)$ is the time-averaged value of velocity gradients that determines the kth moment of the distance. It seems plausible that this value has a power-law behavior in Re similarly to the gradients that determine the usual single-time moments of velocity gradients. Then, since $kq(k)\sim S[q(k)]$ is implied by maximization of $k\rho-S(\rho)$, we find

$$q(k)\propto S[q(k)]\propto\mathrm{Re}^{\zeta(k)},\quad \gamma(k)\tau_\nu=b_k\mathrm{Re}^{\zeta(k)}, \tag{29}$$

where b_k are constants or weak functions of Re, and $\zeta(k)$ is a nontrivial function.

Hyperintermittency

We provide quantitative description of the hyperintermittency considered in the Introduction. The above equalities tell that the volume fraction of spatial regions that determine $\langle r^k\rangle_s$ behaves as $\sim\exp(-t\tilde{b}_k\mathrm{Re}^{\zeta(k)})$ with quasiconstant functions of the Reynolds number $\tilde{b}_k$ (here and below we call functions that are either constants or depend on Re slower than a power-law "quasiconstants" as their role is no different from constants. These functions depend on k though). The regions are exponentially rare in both time and Re. Similarly, the growth of the moment of the distance is exponential both in time and Re. The physics of this growth is that it occurs due to gradients that are larger than a typical value τ_ν^{-1} by a power of Re and are also preserved on average during very long time. This is more described above.

Jensen's inequality $\exp(k\int_0^t\langle\hat{n}\nabla\boldsymbol{v}\hat{n}\rangle dt')\leqslant\langle\exp(k\int_0^t\hat{n}\nabla\boldsymbol{v}\hat{n}dt')\rangle$ implies that $\gamma(k)\geqslant k\lambda_1$. Due to intermittency this inequality is strict. The inequality $\gamma(k)/(k\lambda_1)>1$ is well-known and holds also for Gaussian velocities [13]. However, this intermittency is nonparameteric and $\gamma(k)/(k\lambda_1)\sim 1$ for moderate k. Here we stress that for turbulence the deviation from the self-similar growth of distances that is described by $\gamma(k)=k\lambda_1$ is parametric; i.e., $\gamma(k)/(k\lambda_1)$ is a power of Re with a k-dependent exponent.

D. Strong nonparabolicity of $\gamma(k)$

The results above imply that the very usual quadratic approximation to $\gamma(k)$, e.g., arising in the so-called Kraichnan model [13], is inconsistent with intermittency. Indeed, $\gamma(k)$ obeys the constraint $\gamma(-3)=0$; see Refs. [14,29]. Together with $\gamma'(0)=\lambda_1$ and $\gamma(0)=0$ this constrains the quadratic approximation to the form $\gamma(k)=\lambda_1k(k+3)/3$. This would imply that μ has the same dependence on Re as λ_1 producing inconsistency. The deviation from parabolicity grows indefinitely with Re since, neglecting the weak dependence of $\lambda_1\tau_\nu$ on Re, we have $\mu/\lambda_1 \sim \mathrm{Re}^\kappa$. In contrast, the quartic approximation to $\gamma(k)$, that was demonstrated in Ref. [14] to agree with the available observations, does not have this deficiency.

It could also be thought that quadratic approximation provides a lower bound for $\gamma(k)$ and intermittency implies faster than parabolic (Gaussian) growth,

$$\gamma(k) \geqslant k\lambda_1 + \frac{\mu k^2}{2}, \tag{30}$$

cf. Eq. (21). However, this inequality cannot be true at any $k > 0$. Indeed, the skewness of velocity derivatives is negative which implies that Δ_3 is most probably negative. Then, Eq. (21) implies that the inequality above does not hold for small k. Yet, for large k the inequality must be true because the decay of $P(\rho, t)$ is indeed slower than Gaussian; see Refs. [14] and references therein. Thus, the above inequality is true for k larger than some k_0. Since Δ_4 is positive and, belonging to higher order moment of velocity derivatives, it grows faster with Re than Δ_3, then $k_0 \sim 1$ is probable. The detailed study is left for future work.

E. Resume and numerical study of $\gamma(k)$

The main contribution of this section is providing an approach to the study of the Re dependence of the generalized Lyapunov exponent. This approach relies on the observation that the cumulants have a power-law dependence on Re and represents $\gamma(k)$ as a series in the cumulants. This seems to be of use not only theoretically. The numerical studies of $\gamma(k)$ depends on the accumulation of statistically significant pool of data that contains stretching histories whose fraction decays with time (hyper)exponentially. Our study indicates that the higher Re is the more demanding this becomes. At the same time the well-developed accurate procedures that are used for measuring the Re dependence of equal time moments of velocity gradients may be employed for the numerical study of the cumulants. As of today, the dependence was measured (with varying accuracy) for moments up to 12−th order, see the references in Ref. [21]. Thus, the Re dependence can be obtained for Δ_n with $n \leqslant 12$ for the highest resolution simulations to date. Summation of the first 12 terms in Eq. (21) will give $\gamma(k)$ for a significant range of k.

III. INFINITE-DIMENSIONAL LYAPUNOV EXPONENT: MAGNETIC FIELD

The results of the previous section are manifested most remarkably by considering growth of small fluctuations of magnetic field $\boldsymbol{B}$ in turbulent flows of ideally conducting fluids. The implications are immediate in the Lagrangian frame and can be transferred to the Eulerian frame that is more relevant in this case. The magnetic field obeys the induction equation

$$\partial_t \boldsymbol{B} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{v}, \tag{31}$$

where the flow $\boldsymbol{v}$ is considered to obey the usual Navier-Stokes equations. Thus, we study the kinematic dynamo where $\boldsymbol{B}$ is considered to be small so that the Lorentz force in the momentum equation is negligible. The magnetic field in the particle's frame obeys equation identical to Eq. (3),

$$\frac{d}{dt}\boldsymbol{B}[t, \boldsymbol{q}(t, \boldsymbol{x})] = \{\boldsymbol{B}[t, \boldsymbol{q}(t, \boldsymbol{x})]\cdot\nabla\}\boldsymbol{v}. \tag{32}$$

This equation describes the well-known fact that at zero magnetic diffusivity magnetic field lines are frozen in the fluid and behave as its infinitesimal line elements [17]. We find immediately that

$$\lim_{t\to\infty}\frac{\ln B[t, \boldsymbol{q}(t, \boldsymbol{x})]}{t} = \lambda_1(\boldsymbol{x}), \quad \langle B^k[t, \boldsymbol{q}(t, \boldsymbol{x})]\rangle_s \sim e^{\gamma(k)t},$$

where we assume that $B(t=0, \boldsymbol{x})$ is bounded and omit the initial condition factor in the argument of the logarithm for conciseness. Thus, for almost all $\boldsymbol{x}$ the asymptotic logarithmic growth rate of $B[\boldsymbol{q}(t, \boldsymbol{x}), t]$ is λ_1. We also observe that $\boldsymbol{B}(t, \boldsymbol{x}) = B\hat{n}(t, \boldsymbol{x})$ at $t \gg \tau_\nu$. Indeed, the relaxation to the major stretching direction occurs also for events that involve large deviations of the type described in the previous section, i.e., holds for all relevant events [20].

Incompressibility implies that there is no difference between the Lagrangian and Eulerian averages (in compressible case things are quite different [13,30]). Thus, for the Eulerian average $\langle B^k\rangle_s$ we have

$$\langle B^k(t)\rangle_s \equiv \int B^k(t, \boldsymbol{x}')d\boldsymbol{x}' = \int B^k[t, \boldsymbol{q}(t, \boldsymbol{x})]d\boldsymbol{x} \sim e^{\gamma(k)t}, \tag{33}$$

where $\boldsymbol{x}' = \boldsymbol{q}(t, \boldsymbol{x})$. This equality implies also that the asymptotic logarithmic growth rate at a fixed point in space is λ_1 so that

$$\lim_{t\to\infty}\frac{\ln B(t, \boldsymbol{x})}{t} = \lambda_1, \tag{34}$$

for almost all $\boldsymbol{x}$. To demonstrate the above we observe that

$$\ln\frac{B(t, \boldsymbol{x})}{B(0, \boldsymbol{q}(0|t, \boldsymbol{x}))} = \int_0^t dt'[\hat{n}\nabla\boldsymbol{v}\hat{n}][t', \boldsymbol{q}(t'|t, \boldsymbol{x})], \tag{35}$$

which is a form of Eq. (5), where we use $\boldsymbol{q}(t'|t, \boldsymbol{x})$ introduced in Eq. (25) with $t' < t$. We assume $t \gg \tau_\nu$ in Eq. (35) which allows to neglect the transients to equality $\boldsymbol{B}(t, \boldsymbol{x})/B(t, \boldsymbol{x}) \approx \hat{n}(t, \boldsymbol{x})$. We find dividing Eq. (35) by t and letting $t \to \infty$ that by the multiplicative ergodic theorem the limit is constant for almost all $\boldsymbol{x}$. This constant can be found by spatial averaging of Eq. (35). Transformation of the integration variable from $\boldsymbol{x}$ to $\boldsymbol{q}(t'|t, \boldsymbol{x})$ reproduces the form of λ_1 given by Eq. (8) yielding Eq. (34). The same conclusion is implied by $\langle \ln B\rangle/t = \lambda_1$ obtained by differentiating Eq. (33), setting $k = 0$ and using $\gamma'(0) = \lambda_1$. Yet another proof will be given below.

A. Nonuniform convergence to λ_1

The description of the spatial distribution of the exponentially growing field $\boldsymbol{B}(t, \boldsymbol{x})$ can be transferred from that of $r(t, \boldsymbol{x})$ and is briefly provided here. The asymptotic in time growth of the magnetic field at the same rate almost everywhere in no way implies that there is ever a time where $\ln B(\boldsymbol{x}, t)/t \approx \lambda_1$ is a good approximation uniformly in space. For instance, no matter how large t is, the estimate $\int B^2(t, \boldsymbol{x})d\boldsymbol{x} \sim \exp(2\lambda_1 t)$ does not provide a viable approximation for the magnetic energy whose true value is exponentially larger and is given by $\exp[\gamma(2)t]$. The reason is intermittency. That causes an exponentially decaying in time fraction of the total volume, where the growth rate of the magnetic field deviates significantly from λ_1, to carry most of the energy. The time $t^*(\boldsymbol{x})$ starting from which $\ln B(t,\boldsymbol{x})/t \approx \lambda_1$ holds, strongly depends on $\boldsymbol{x}$. We introduce the exponential growth field θ, which is similar to ρ above, by

$$B(t,\boldsymbol{x}) = \exp\left[\theta(t,\boldsymbol{x})t\right], \quad \theta(t,\boldsymbol{x}) \equiv \frac{\ln B(t,\boldsymbol{x})}{t}. \qquad (36)$$

Thus, $\theta(t,\boldsymbol{x})$ is the growth exponent of the magnetic field at the fixed point in space $\boldsymbol{x}$. Its spatial fluctuations are described by the same large deviations theory as ρ,

$$P(\theta,t) \sim e^{-tS(\theta)}, \quad \langle B^k(t)\rangle \sim \int e^{t[k\theta - S(\theta)]}d\theta, \qquad (37)$$

where S is the same function that we introduced in the previous section due to equality of Eulerian and Lagrangian averages. This probability density function (PDF) implies as previously that $\lim_{t\to\infty} P(\theta,t) = \delta(\theta-\lambda_1)$ providing another proof of Eq. (34). Quadratic expansion of $S(\theta)$ near the minimum at λ_1 produces the central limit theorem [10,11,13]. The probability distribution function of $\{\ln B[\boldsymbol{q}(t,\boldsymbol{x}),t] - \lambda_1 t\}/\sqrt{t}$ (and thus also of $[\ln B(\boldsymbol{x},t) - \lambda_1 t]/\sqrt{t}$) becomes Gaussian in the infinite time limit with dispersion μ [similar conclusion holds for $r(t,\boldsymbol{x})$].

The most relevant of the exponents $\gamma(k)$, described by Eq. (29), in this case is $\gamma(2)$. It describes the growth rate of the magnetic energy and thus determines when the neglect of the magnetic component of the full energy of the conducting fluid becomes inconsistent. The previous results imply that the energy is concentrated in exponentially rare spatial regions whose volume fraction decays as $\exp\{-S[q(2)]t\}$ where $S[q(2)] \propto \mathrm{Re}^{\zeta_2}$. This hyperintermittency implies that accurate measurement of the growing magnetic energy is an exceedingly difficult task at high Re. The estimate of the growth rate of energy by τ_ν^{-1} fails completely at large Re. Similar conclusions hold for other moments of the magnetic field.

B. Field view

The studied case provides a bridge between finite and infinite-dimensional systems that will be useful below. Indeed the induction equation, given by Eq. (31), written in the form $\partial_t \boldsymbol{B} = \hat{L}(t)\boldsymbol{B}$ where $\hat{L}(t)$ is a linear operator, can be considered as infinite-dimensional generalization of the equation on separation written in the form $\dot{r}_i = L_{ik}(t)r_k$ with $L_{ik} \equiv \nabla_k v_i$, see Eq. (3). We observe that trivially, in a finite-dimensional system we can use any of l_k norms in the definition of the Lyapunov exponents that is

$$\lambda_1 = \lim_{t\to\infty} \frac{\ln\left(||\boldsymbol{r}(t)||_k/||\boldsymbol{r}(0)||_k\right)}{t},$$
$$||\boldsymbol{r}(t)||_k = \left(\sum_{i=1}^{3} |r_i(t)|^k\right)^{1/k}. \qquad (38)$$

It is readily seen that λ_1 defined by the above equation is independent of k, since the limit implies that the maximal component of $\boldsymbol{r}$ grows at the rate λ_1; see Sec. I. In contrast, the infinite-dimensional counterpart of the above equation,

$$\lim_{t\to\infty} \frac{\ln\left(||\boldsymbol{B}(t)||_k/||\boldsymbol{B}(0)||_k\right)}{t} = \frac{\gamma(k)}{k},$$
$$||\boldsymbol{B}(t)||_k \equiv \left(\int B^k(t,\boldsymbol{x})d\boldsymbol{x}\right)^{1/k}, \qquad (39)$$

depends on k. This gives a more solid foundation to the name "generalized Lyapunov exponent" since $\gamma(k)/k$ is proportional to the true infinite-dimensional Lyapunov exponent defined with the help of the L_k norm. Thus, our assumption that $\gamma(k)$ is independent of realization of velocity is equivalent to the assumption of realization-independence of infinite-dimensional Lyapunov exponents defined with the corresponding norm.

C. Hyperintermittency law

Finally, we give another outlook at the results of this section which will be useful below. It is found by differentiating Eq. (31) that

$$\frac{d}{dt}\int B^{2k}(t,\boldsymbol{x})d\boldsymbol{x} = 2k\int B^{2k-2}B_i B_k \nabla_i v_k d\boldsymbol{x}, \qquad (40)$$

where we assume that the boundary term can be neglected or is zero as in the case of periodic or zero normal velocity boundary conditions. We find, recalling the alignment of $\boldsymbol{B}$ with the major stretching direction, that

$$\frac{d}{dt}\ln\int B^{2k}(t,\boldsymbol{x})d\boldsymbol{x} = 2k\langle \hat{n}\nabla \boldsymbol{v}\hat{n}\rangle_{2k}, \qquad (41)$$

where we defined the average weighted with normalized measure $B^{2k}(t,\boldsymbol{x})/\int B^{2k}d\boldsymbol{x}$ by

$$\langle \hat{n}\nabla \boldsymbol{v}\hat{n}\rangle_{2k} \equiv \frac{\int B^{2k}\hat{n}\nabla \boldsymbol{v}\hat{n}d\boldsymbol{x}}{\int B^{2k}(t,\boldsymbol{x})d\boldsymbol{x}}. \qquad (42)$$

We obtain from the above by using Eq. (29) and definition of $\gamma(k)$ that

$$\langle \hat{n}\nabla \boldsymbol{v}\hat{n}\rangle_k = \frac{\gamma(k)}{k} = \frac{b_k \mathrm{Re}^{\zeta(k)}}{k\tau_\nu}. \qquad (43)$$

The last identities provide robust description of the hyperintermittency of the growth. The normalized measure proportional to B^{2k} is concentrated in the regions where the gradients are larger than inverse Kolmogorov time by a power of Re. Conversely, the $2k$th moment of the field, which includes the energy, grows in regions where due to intermittency the gradients are larger than τ_ν^{-1} by $\mathrm{Re}^{\zeta(k)}$.

IV. FINITE MAGNETIC DIFFUSIVITY

In this section we demonstrate that the Re dependence of the growth of the magnetic field in an ideally conducting fluid, derived previously, generalizes qualitatively to the case of finite magnetic diffusivity η. The main reason is that since the field growth at $\eta \ll \nu$ occurs mainly on spatial scales smaller than the Kolmogorov scale and larger than the diffusive scale [19], it may be described with the help of finite-time Lyapunov exponents [10,11,13]. The latter do not differ qualitatively from the previously considered ρ variable; see Ref. [14] for relation between the exponents. Here we refer to previous works on the case of small diffusivity, reconsidering their formulas from the viewpoint of their Reynolds number dependence, which was not done previously.

The ideal fluid approximation studied in the previous section has a limited domain of validity. It is valid when the magnetic diffusivity η that appears in the full induction equation [17],

$$\partial_t \boldsymbol{B} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{B} = (\boldsymbol{B}\cdot\nabla)\boldsymbol{v} + \eta\nabla^2\boldsymbol{B}, \tag{44}$$

is much smaller than ν, cf. Eq. (31). This condition is usually referred to as the condition of large magnetic Prandtl number $\Pr \equiv \nu/\eta$. If $\Pr \gg 1$ then there is a finite interval of time during which the smallest spatial scale of the magnetic field is not too small and the diffusivity term is negligible everywhere. During such interval the assumption of ideal fluid applies. However, the mixing property of turbulence generates increasingly smaller scales unless the diffusivity term with higher order derivative becomes relevant for any value of η regardless of how small it is [17]. The resulting impact of η on the growth exponents

$$\gamma^B(k,\eta) \equiv \lim_{t\to\infty}\frac{1}{t}\ln\left[\frac{\int B^k(t,\boldsymbol{x})d\boldsymbol{x}}{\int B^k(0,\boldsymbol{x})d\boldsymbol{x}}\right], \tag{45}$$

is thus finite for $\eta > 0$. We demonstrated previously that at $\eta = 0$ it is highly plausible that the limits above exist and do not depend on the flow realization or the initial conditions $\boldsymbol{B}(0,\boldsymbol{x})$. It seems that finite magnetic diffusivity, that enhances the mixing of $\boldsymbol{B}$, seemingly can only strengthen the independence of the limit of the flow realization. Thus, we assume below that $\gamma^B(k,\eta)$ are well-defined realization-independent properties.

The realization-independence can actually be proven by using the formula for B^2 obtained in Ref. [19] in the case of $\Pr \gg 1$ and not too large times, see below. Thus, Ref. [19] provides B^2 as exponential of linear combinations of finite-time Lyapunov exponents. These exponents are similar to the ρ variable above, which is also a type of finite-time Lyapunov exponent, see definition and description in Refs. [10,11,13,14]. The application of the cumulant expansion theorem to the spatial average then demonstrates the existence and realization-independence of $\gamma^B(k,\eta)$ quite similarly to the demonstration for $\gamma(k)$.

The formula for $\gamma^B(k,\eta)$ was obtained in Ref. [19]. This work can be used to demonstrate that despite that $\gamma^B(k,\eta)$ is discontinuous at $\eta = 0$ the nature of the Re dependence of the growth is unchanged. To see that, we observe that as is demonstrated in Ref. [19], the pointwise growth of the magnetic field is given by

$$\lim_{t\to\infty}\frac{\ln B(t,\boldsymbol{x})}{t} = \min\left(\frac{\lambda_1-\lambda_2}{2},\frac{\lambda_2-\lambda_3}{2}\right),\quad \Pr\gg 1, \tag{46}$$

which in the limit $\eta\to 0$ is clearly different than the corresponding value at $\eta = 0$, cf. Eq. (34). Since the Re dependence of the exponents is identical, see above, then we find

$$\lim_{t\to\infty}\frac{\ln B(t,\boldsymbol{x})}{t} = \frac{d\gamma_B}{dk}(k=0) = \frac{\tilde{c}_0}{\tau_\nu}\mathrm{Re}^{-\delta}, \tag{47}$$

where δ is the same exponent that appears in Eq. (17) and $\tilde{c}_0$ is quasiconstant. Thus, despite the discontinuity in the growth rate at $\eta = 0$, its dependence on Re is the same for $\eta \to 0$ and $\eta = 0$ (the constant $\tilde{c}_0$ is smaller than its counterpart at $\eta = 0$).

The kth moment of the magnetic field can be found as the average of an exponent containing linear combinations of finite-time Lyapunov exponents [19]. The exponents obey statistics similar to that of ρ above [10,13]. The result of the averaging is therefore similar,

$$\gamma^B(k)\tau_\nu = \tilde{b}_k \mathrm{Re}^{\tilde{\zeta}(k)},\quad \eta\to 0, \tag{48}$$

where the scaling exponents $\tilde{\zeta}(k)$ and quasiconstants $\tilde{b}_k$ generally differ from $\zeta(k)$ and b_k in Eq. (29) except for $\tilde{\zeta}(0) = \zeta(0)$. Generally, a strict inequality is realized in $\gamma^B(k,\eta\to 0) \leqslant \gamma^B(k,\eta=0)$ implying $\tilde{\zeta}(k) < \zeta(k)$.

A possible difference between $\gamma^B(k)$ and $\gamma(k)$ is their analytic properties. The function $\gamma(k)$ has continuous derivatives and might well be analytic; see Eq. (21). In contrast, $\gamma^B(k)$ might not be analytic. Changes of regime producing discontinuities in the first derivative of $\gamma^B(k)$ are possible similar to Ref. [10] where for some k the average is determined by the boundary of the range of variations of the finite-time Lyapunov exponents and for some by the interior. The existence of points with discontinuous first derivative depends on the form of the large deviations function of the finite time Lyapunov exponents and its study is beyond our scope here. In Kraichnan model $\gamma^B(k)$ was found to be analytic [19].

The properties of averaging over finite-time Lyapunov exponents are similar to those of averaging over ρ. Exponential growth of the moments $\langle B^k(t)\rangle \sim \exp(\gamma^B(k)t)$ implies that the PDF of $\theta \equiv \ln B/t$ still has the large deviations form given by Eq. (37). The large deviations function $\tilde{S}(\theta)$ is the Legendre transform of $\gamma^B(k)$ that has minimal value zero attained at the argument equal to the RHS of Eq. (46). Thus, Eq. (46) holds with probability one which implies that it holds also independently of $\boldsymbol{B}(t=0)$ and realization of velocity. The volume fraction of the regions that determine a given moment is again exponentially small both in time and in the Reynolds number as in the case of $\eta = 0$. The detailed derivation of above properties from Ref. [19] is beyond the scope of this paper and will be published elsewhere.

A. Hyperintermittency law at $\eta > 0$

We can reinterpret the above results in terms of a hyperintermittency law similar to that in the previous section. We observe that

$$\frac{d}{dt}\int B^{2k}(t,\boldsymbol{x})d\boldsymbol{x} = 2k\int d\boldsymbol{x}(B_iB_k\nabla_i v_k + \eta B_i\nabla^2 B_i)\times B^{2k-2}(t,\boldsymbol{x}) \sim 2k\int B^{2k-2}B_iB_k\nabla_i v_k d\boldsymbol{x}; \tag{49}$$

cf. Eq. (40). The last asymptotic equality recognizes that the growth occurs when the magnetic lines' stretching and diffusivity terms are of the same order (otherwise, either the growth exponents were as in $\eta = 0$ case or the diffusivity term would dominate and stop the growth). We find that

$$\frac{2k\int B^{2k-2}B_iB_k\nabla_i v_k d\boldsymbol{x}}{\int B^{2k}(t,\boldsymbol{x})d\boldsymbol{x}} \sim \gamma^B(k) = \frac{\tilde{b}_k}{\tau_\nu}\mathrm{Re}^{\tilde{\zeta}(k)}. \tag{50}$$

This implies that qualitatively, as in $\eta = 0$ case, the growth of the $2k-$th moment occurs due to gradients that are larger than τ_ν^{-1} by a power of Re. The geometry of magnetic field configurations changes and the growth exponents are depleted by factor of order one [19], yet the Re dependence is similar to $\eta = 0$ case.

The results of Ref. [19] hold for times large enough that the diffusivity is relevant at the smallest scale of $\boldsymbol{B}(t, \boldsymbol{x})$, yet small enough that the linear size of the zone of dependence of $\boldsymbol{B}(t, \boldsymbol{x})$ [that is the spatial region where changes of the initial conditions $\boldsymbol{B}(t = 0)$ change the value of $\boldsymbol{B}(t, \boldsymbol{x})$ appreciably] is much smaller than the viscous scale l_ν. The last assumption is necessary in order to describe the velocity by linear profile as done in that study. Therefore, Ref. [19] does not allow to study the limit $t \to \infty$ being confined, as can be seen, to time interval of order $\tau_\nu \ln \mathrm{Pr}$ (correspondingly the infinite time limits in Eqs. (46) and (47) must be understood in asymptotic sense). Beyond these times the zone of dependence is larger than l_ν. During the studied time interval of order $\tau_\nu \ln \mathrm{Pr}$ exponentially growing moments of the magnetic field, including the energy, increase by a power of Pr.

The study of $t \gg \tau_\nu \ln \mathrm{Pr}$, that ultimately fixes $\gamma^B(k, \eta)$ in Eq. (45), requires other techniques. For finding the energy exponent $\gamma^B(2)$ one can use the evolution equation for the spectrum of magnetic field [31]; see also Ref. [32]. That equation is formulated in terms of finite-time Lyapunov exponents and hence the result of the solution will still have the form given by Eq. (48); see also Ref. [33]. For other moments a different scheme needs to be developed where probably the most intriguing question is what comes instead of Eq. (46). These studies are left for future work. Here we observe that there seem to be no reason for a qualitative change of Eq. (50) because the magnetic field is concentrated at a diffusive scale that is much smaller than the viscous scale and where the linear velocity approximation is valid. Decrease of Pr leads to increase of diffusive scale that becomes equal to the viscous scale at $\mathrm{Pr} = 1$. The asymptotic continuation then indicates that

$$\gamma^B(k, \eta)\tau_\nu = \tilde{b}_k(\eta)\mathrm{Re}^{\tilde{\zeta}(k,\eta)}, \quad \mathrm{Pr} \lesssim 1, \tag{51}$$

with $\tilde{b}_k(\eta)$ and $\tilde{\zeta}(k, \eta)$ smooth functions of η. Thus, the growth of the magnetic field at $\mathrm{Pr} = 1$, as governed by

$$\partial_t \boldsymbol{B} + (\boldsymbol{v} \cdot \nabla)\boldsymbol{B} = (\boldsymbol{B} \cdot \nabla)\boldsymbol{v} + \nu\nabla^2\boldsymbol{B}, \tag{52}$$

must still exhibit Re-dependent exponents. This agrees qualitatively with observations considered in the next section. The law given by Eq. (48) is valid also at $\mathrm{Pr} \ll 1$ with magnetic Reynolds number taking the place of Re as will be reported in a successive publication.

To summarize, the limit of zero magnetic diffusivity is singular, but the results at this limit are still useful. It seems that it preserves the power-law dependence on Re of the growth exponents of the magnetic field moments that has been deduced for $\eta = 0$. This can be proved rigorously in the intermediate time regime studied by Ref. [19]. This can also be proved at $t \to \infty$ for $\gamma_B(2)$ using the spectrum equation of Ref. [31]. The full study is left for future work. Finally the results at $\mathrm{Pr} \gg 1$ seem to extend to $\mathrm{Pr} \sim 1$.

V. Re DEPENDENCE OF GENERALIZED LYAPUNOV EXPONENTS OF TURBULENCE

In this section we introduce the generalized Lyapunov exponent of turbulence which describes the growth rate of different moments of the velocity perturbation. Probably the main contribution of this section to the current research is raising of the possibility that a pointwise logarithmic growth rate exists that holds almost everywhere at large times. This rate differs strongly from the quantity which is currently called the Lyapunov exponent of turbulence.

The generalized Lyapunov exponent of turbulence describes exponential growth of separation $\delta\boldsymbol{v}$ of two solutions of the NS equations given by Eqs. (1). The evolution of $\delta\boldsymbol{v}$ is described by the linearized NS equations

$$\partial_t\delta\boldsymbol{v} + (\boldsymbol{v} \cdot \nabla)\delta\boldsymbol{v} = -(\delta\boldsymbol{v} \cdot \nabla)\boldsymbol{v} - \nabla\delta p + \nu\nabla^2\delta\boldsymbol{v}, \tag{53}$$

and the incompressibility condition $\nabla \cdot \delta\boldsymbol{v} = 0$. The equation's difference from Eq. (52) consists of the different sign of the lines' stretching term and the presence of the pressure which is necessary to ensure solenoidality. These changes would not change the exponential growth of $\delta\boldsymbol{v}$ similar to $\boldsymbol{B}$. The issue is how we measure this growth. The usually used definition of the Lyapunov exponent of turbulence is

$$\lambda^v \equiv \lim_{t\to\infty} \frac{1}{t} \ln \left\{ \frac{\left[\int (\delta v)^2(\boldsymbol{x}, t) d\boldsymbol{x}\right]^{1/2}}{\left[\int (\delta v)^2(\boldsymbol{x}, 0) d\boldsymbol{x}\right]^{1/2}} \right\}; \tag{54}$$

see, e.g., Refs. [4,5,8,34]. It is, however, obvious from the previous studies that the preference for the L_2 norm is not self-evident and a more versatile characteristics of the growth is provided by the generalized Lyapunov exponent $\gamma^v(p)$ defined in the Introduction

$$\gamma^v(p) \equiv \lim_{t\to\infty} \frac{1}{t} \ln \left\{ \frac{\left[\int |\delta v|^p(t, \boldsymbol{x}) d\boldsymbol{x}\right]^{1/p}}{\left[\int |\delta v|^p(0, \boldsymbol{x}) d\boldsymbol{x}\right]^{1/p}} \right\}, \tag{55}$$

where $\lambda^v = \gamma^v(2)$. We see that for any p we can interpret $\gamma^v(p)$ as “the” Lyapunov exponent of the NS equation should we decide to declare L_p as “the” norm on the functional space. Thus, $\gamma^v(p)$ are probably realization-independent quantities which is reinforced by their similarity with $\gamma^B(p, \eta = \nu)/p$ considered in the previous section. More generally there is qualitative similarity between $\gamma(k)$, $\gamma^B(k)$ and $k\gamma^v(k)$. Thus, λ^v must behave as $\gamma(2)$ and has a power-law-type growth with Re as indeed observed in Ref. [5]. This fully explains the discrepancy of the observed behavior of λ^v and that predicted in Ref. [8], see the qualitative explanation in the Introduction.

The above analogy would indicate the probable validity of

$$\langle|\delta v|^k\rangle_s \sim \exp(\gamma^v(k)t), \quad \gamma^v(k)\tau = d(k)\mathrm{Re}^{\xi(k)}, \tag{56}$$

where $d(k)$ are quasiconstants. This form implies, similarly to our previous studies, that the PDF of $\phi(t, \boldsymbol{x}) \equiv t^{-1}\ln|\delta\boldsymbol{v}(t, \boldsymbol{x})|$ has large deviations form $\sim\exp[-tH(\phi)]$. Here H is the Legendre transform of $\gamma^v(k)$ that has unique minimum of zero at $\phi = \lambda_0$ where $\lambda_0 \equiv d\gamma^v/dk(k = 0)$. We conclude that

$$\lim_{t\to\infty} \frac{\ln|\delta\boldsymbol{v}(t, \boldsymbol{x})|}{t} = \lambda_0, \tag{57}$$

holds almost everywhere for almost every perturbed flow $\boldsymbol{v}(t, \boldsymbol{x})$ and almost any initial perturbation field $\delta\boldsymbol{v}(t, \boldsymbol{x})$. The above considerations indicate that it is plausible that $\lambda_0\tau_\nu$ is a weakly decaying function of Re. Strictly speaking the above considerations are a hypothesis which detailed study is left for future work.

VI. CONCLUSIONS

In this work we established some properties of three sets of Lyapunov exponents associated with the Navier-Stokes turbulence. The first set $\gamma(k)$ describes the exponential divergence of two infinitesimally close Lagrangian trajectories of fluid particles. The second set $\gamma^B(k, \eta)$ is that of magnetic kinematic dynamo, the set which depends on the magnetic diffusivity η. Finally the third set $\gamma^v(k)$ provides infinite set of possible definitions of the Lyapunov exponent of the Navier-Stokes equations distinguished by the different definitions of the norm that is assigned to the flow perturbation field. It seems that this set has never been considered before, with the exception of the L_2 norm Lyapunov exponent.

We demonstrated using cumulant expansion theorem that $\gamma(k)$, defined by spatial averaging, is independent of the velocity realization and is given by a series whose terms are growing powers of Re. Using large deviations theory we demonstrated that $\gamma(k)$ plausibly obeys power-law behavior in Re which agrees with all current numerical data. We demonstrated that the growth is hyperintermittent. A qualitative picture of the latter emerges by considering the flow regions where velocity gradients are anomalously large, exceeding the typical value of τ_ν^{-1} by a power of Re. In those regions the stretching of an infinitesimal line element that connects the trajectories' positions occurs much faster than in typical regions with gradients of order τ_ν^{-1}. Moreover, there are highly anomalous fluctuations where these large gradients, on (time) average, persist along the Lagrangian trajectory during an arbitrarily long time interval t. Since correlation time of these large gradients is finite then the probability of these persistent in time large fluctuations decays exponentially in time according to extremely rapid decay law $\sim\exp(-ct\text{Re}^\alpha/\tau_\nu)$ with some constant c and exponent α. Still these events are not negligible because stretching of line elements on these events is extremely fast. The stretching rate is roughly constant and proportional to Re^β with some exponent β so that the distance grows proportionally to $\exp(c' t\text{Re}^\beta/\tau_\nu)$ where c' is a constant. As a result these events determine the growth rate of the moments since extreme smallness of the probability of the event is compensated by extreme largeness of averaged quantity on these events.

The Re dependence of the exponents implies that care is needed in picking the numerical resolution of simulations to measure them, more so as higher Re are considered. For the Lyapunov exponent of the fluid particles, which is decreased by intermittency in comparison with the Kolmogorov theory estimate, grid resolution with size of order of Kolmogorov scale will do since that is determined by the calm long-correlated turbulent events. In contrast, the Lyapunov exponent of the Navier-Stokes equations or the generalized Lyapunov exponent of the fluid particles, which are increased by intermittency in comparison with the Kolmogorov theory estimate, are determined by intermittent bursts whose scale is possibly much smaller than the Kolmogorov scale, cf. the study of dependence on lattice size in Ref. [16].

The Re dependence would hold also for many other similar quantities. This includes the counterpart of the generalized Lyapunov exponent for the vorticity stretching introduced in Ref. [35], the growth rate of moments of concentration of inertial particles in turbulence [30], the generalized Lyapunov exponent of magnetohydrodynamic turbulence [16], and others.

Theory of the Lyapunov exponent of the Navier-Stokes equations demands insight into the fastest growing perturbations. These were reported to be correlated with helicity [36] however the considered Re are too small to make a conclusion on the infinite Re limit. The reference studies the full spectrum of the Lyapunov exponents of the Navier-Stokes equations. It seems that Re dependence of different exponents is different. This implies that there could be simplifications in the Kaplan-Yorke dimension of turbulence at $\text{Re} \to \infty$.

It is quite probable that there exists the generalization of multiplicative large deviations theory, as described by statistics of finite-time Lyapunov exponents to the case of infinite-dimensional operators such those that govern the growth of small perturbations of the Navier-Stokes equations or small fluctuations of magnetic field in turbulence of conducting fluids. The development of the corresponding formalism is a worthy research venue.

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